THE LOCAL FORM OF DOUBLY STOCHASTIC MAPS AND JOINT MAJORIZATION IN II_1 FACTORS

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Dedicated to our families

Abstract. We find a description of the restriction of doubly stochastic maps to separable abelian C^*-subalgebras of a II_1 factor $\mathcal{M}$. We use this local form of doubly stochastic maps to develop a notion of joint majorization between n-tuples of mutually commuting self-adjoint operators that extends those of Kamei (for single self-adjoint operators) and Hiai (for single normal operators) in the II_1 factor case. Several characterizations of this joint majorization are obtained. As a byproduct we prove that any separable abelian C^*-subalgebra of $\mathcal{M}$ can be embedded into a separable abelian C^*-subalgebra of $\mathcal{M}$ with diffuse spectral measure.

1. Introduction

Majorization between self-adjoint operators in finite factors was introduced by Kamei [18] as an extension of Ando's definition of majorization between self-adjoint matrices [4], a useful tool in matrix theory. Later on, Hiai considered majorization in semifinite factors between self-adjoint and normal operators [12, 13]. The reason why majorization has attracted the attention of many researchers (see the discussion in [13] and the references therein) is that it provides a rather subtle way to compare operators and occurs naturally in many contexts (for example [5, 10, 11]). Recently, majorization has regained interest because of its relation with norm-closed unitary orbits of self-adjoint operators and conditional expectations onto abelian subalgebras [5, 6, 8, 11, 15, 16, 20, 22]. One of the goals of this paper (section 4) is to obtain an extension of the notion of majorization between normal operators to that of *joint majorization* between n-tuples of commuting self-adjoint operators in a II_1 factor (such extension is achieved in [19] for finite dimensional factors). In order to obtain characterizations of this extended notion we describe the *local form* of a doubly stochastic map (DS), i.e. we get a family of particularly well behaved DS maps that approximate the restriction of any DS map to separable abelian C^*-subalgebras of a II_1 factor (section 3). As a byproduct, we construct separable abelian diffuse refinements of separable abelian C^*-subalgebras of a II_1 factor $\mathcal{M}$. This construction seems to have interest on its own. Some of the techniques we use seem to be new, even in the single element case.

So far we have restricted our attention to the II_1 factor case because, on one hand, technical aspects of the work become simpler and on the other hand, this is the context where majorization has its full meaning. Since every finite von Neumann algebra acting on a separable Hilbert space has a direct integral decomposition in terms of finite factors, the study of II_1 factors provides useful information about more general algebras.

The paper is organized as follows. In section 2 we recall some facts about abelian C^*-subalgebras of a II_1 factor. In section 3, after describing some technical results, we obtain a description of the local structure of doubly stochastic maps. In section 4 we introduce and develop the notion of joint majorization between finite abelian families of self-adjoint operators in a II_1 factor and we obtain several characterizations of this relation. Finally, in section 6 we prove the results described in section 3.

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2. Preliminaries

Throughout the paper $\mathcal{M}$ will be a II_1 factor with normalized faithful normal trace τ. The C^*-subalgebras of $\mathcal{M}$ are always assumed unital. The subspace of self-adjoint elements of $\mathcal{M}$ will be denoted by $\mathcal{M}_{sa}$, and we will consider **abelian families** $(a_1,\ldots,a_n)=(a_i)_{i=1}^n$ in $\mathcal{M}_{sa}$, that is finite families of mutually commuting self-adjoint operators in $\mathcal{M}$. If $(a_i)_{i=1}^n\subseteq\mathcal{M}_{sa}$ is an abelian family then $C^*(a_1,\ldots,a_n)$ denotes the (unital) separable abelian C^*-subalgebra of $\mathcal{M}$ generated by the elements of the family. If $\mathcal{A}$ is an arbitrary abelian C^*-subalgebra of $\mathcal{M}$ then $\Gamma(\mathcal{A})$ denotes its space of characters, i.e. the set of *-homomorphisms $\gamma:\mathcal{A}\to\mathbb{C}$, endowed with the weak*-topology. The set $\Gamma(\mathcal{A})$ is a compact space and $\mathcal{A}\simeq C(\Gamma(\mathcal{A}))$, where $C(\Gamma(\mathcal{A}))$ denotes the C^*-algebra of continuous functions on $\Gamma(\mathcal{A})$.

2.1. **Joint spectral measures and joint spectral distributions.** As we will consider a several-variable version of functional calculus, we state a few facts about it (see [23] for a different description). Let $\bar a=(a_i)_{i=1}^n$ be an abelian family in $\mathcal{M}_{sa}$. If $\mathcal{A}=C^*(a_1,\ldots,a_n)$, then $\Gamma(\mathcal{A})$ can be embedded in $\prod_{i=1}^n\sigma(a_i)\subseteq\mathbb{R}^n$. In fact, the map $\Phi:\Gamma(\mathcal{A})\to\prod_{i=1}^n\sigma(a_i)\subseteq\mathbb{R}^n$ given by $\Phi(\gamma)=(\gamma(a_1),\ldots,\gamma(a_n))$ is a continuous injection and therefore $\Gamma(\mathcal{A})$ is homeomorphic to its image under this map; this image is called the **joint spectrum** of the family and we denote it by $\sigma(\bar a)\subseteq\prod_{i=1}^n\sigma(a_i)$. Note that $\mathcal{A}\simeq C(\sigma(\bar a))$ as C^*-algebras. If $f\in C(\sigma(\bar a))$, there exists a normal operator, denoted $f(a_1,\ldots,a_n)$, that corresponds to f under the isomorphism $\mathcal{A}\simeq C(\sigma(\bar a))$. This association extends the usual one variable functional calculus.

If $\mathcal{A}\subseteq\mathcal{M}$ is a separable C^*-subalgebra then $\Gamma(\mathcal{A})$ is metrizable and the representation $C(\Gamma(\mathcal{A}))\simeq\mathcal{A}\subseteq\mathcal{M}$ induces a spectral measure $E_{\mathcal{A}}$ [9, IX.1.14] that takes values on the lattice $\mathcal{P}(\mathcal{M})$ of projections of $\mathcal{M}$. Let $\mu_{\mathcal{A}}$ be the (scalar) regular Borel measure on $\Gamma(\mathcal{A})$ defined by

$$\mu_{\mathcal{A}}(\Delta)=\tau(E_{\mathcal{A}}(\Delta)).$$

The regularity of $\mu_{\mathcal{A}}$ follows from the fact that every open set is σ-compact [21, 2.18]. The map $\Lambda:L^\infty(\Gamma(\mathcal{A}),\mu_{\mathcal{A}})\to\mathcal{M}$ given by $\Lambda(h)=\int_{\Gamma(\mathcal{A})}h\,dE_{\mathcal{A}}$ is a normal *-monomorphism (note that in this case the weak* topology of $L^\infty(\Gamma(\mathcal{A}),\mu_{\mathcal{A}})$, restricted to the unit ball, is metrizable) and we have

$$\tau\left(\Lambda(h)\right)=\int_{\Gamma(\mathcal{A})}h\,d\mu_{\mathcal{A}},\quad\forall h\in L^\infty(\Gamma(\mathcal{A}),\mu_{\mathcal{A}}).\tag{1}$$

We consider the von Neumann algebra $L^\infty(\mathcal{A}):=\Lambda(L^\infty(\Gamma(\mathcal{A}),\mu_{\mathcal{A}}))\subseteq\mathcal{M}$.

When $\mathcal{A}=C^*(a_1,\ldots,a_n)$, $E_{\bar a}:=E_{\mathcal{A}}$ and $\mu_{\bar a}:=\mu_{\mathcal{A}}$ are the **joint spectral measure** and **joint spectral distribution** of the abelian family $\bar a$ and we denote by $\Lambda_{\bar a}:L^\infty(\Gamma(\bar a),\mu_{\bar a})\to L^\infty(\mathcal{A})$ the normal isomorphism defined above. It is straightforward to verify that $\Lambda_{\bar a}(\pi_i)=a_i$, $1\le i\le n$, and we write $h(a_1,\ldots,a_n):=\Lambda_{\bar a}(h)$. In the case of a single self-adjoint operator $a\in\mathcal{M}_{sa}$ the measure μ_a is the usual spectral distribution of a (see [8]), and it agrees with the Brown measure of a.

In the particular case when $x\in\mathcal{M}$ is a normal operator, the real and imaginary parts of x are mutually commuting self-adjoint elements of $\mathcal{M}$. Identifying the complex plane with $\mathbb{R}^2$ in the usual way, it is easy to see that the spectrum of x as a normal operator coincides with the joint spectrum of the abelian pair $(\mathrm{Re}(x),\mathrm{Im}(x))$, and that the spectral measure of x coincides with the joint spectral measure of $(\mathrm{Re}(x),\mathrm{Im}(x))$.

2.2. **Comparison of measures and diffuse measures.** We denote by $M_+^\sim(\mathbb{R}^n)$ the set of all regular finite positive Borel measures ν on $\mathbb{R}^n$ with $\int\|\zeta\|\,d\nu(\zeta)<\infty$. We write $\nu(f)=\int_{\mathbb{R}^n}f\,d\nu$, for every $\nu\in M_+^\sim(\mathbb{R}^n)$ and every ν-integrable function f. In what follows, 1 denotes the constant function and $\pi_i:\mathbb{R}^n\to\mathbb{R}$ denotes the projection onto the i^{th} coordinate.

Definition 2.1. *We say that μ is majorized by ν, and we write $\mu\prec\nu$, if for every $\mu_1,\ldots,\mu_m\in M_+^\sim(\mathbb{R}^n)$ with $\sum_{i=1}^m\mu_i=\mu$ there exist $\nu_1,\ldots,\nu_m\in M_+^\sim(\mathbb{R}^n)$ such that $\sum_{i=1}^m\nu_i=\nu$, $\nu_i(1)=\mu_i(1)$ and $\nu_i(\pi_j)=\mu_i(\pi_j)$ for $1\le i\le m$, $1\le j\le n$.*

The relation $\prec$ in Definition 2.1 does not seem to be called "majorization" in the literature, but it will be a suitable name for us in the light of Theorem 4.5. If $\mu,\, \nu \in M_+^{\sim}(\mathbb{R}^n)$ we shall write $\nu \sim \mu$ whenever $\nu(1) = \mu(1)$ and $\nu(\pi_j) = \mu(\pi_j)$ for every $1 \le j \le n$.

Theorem 2.2. [3, I.3.2] *Let $\mu,\, \nu \in M_+^{\sim}(\mathbb{R}^n)$. Then $\mu \prec \nu$ if and only if $\mu(f) \le \nu(f)$ for every continuous convex function $f : \mathbb{R}^n \to \mathbb{R}$.*

The next corollary is a direct consequence of Theorem 2.2 and the identity in equation (1).

Corollary 2.3. *Let $\bar a = (a_i)_{i=1}^n$, $\bar b = (b_i)_{i=1}^n \subset \mathcal{M}_{sa}$ be two abelian families. Then $\mu_{\bar a} \prec \mu_{\bar b}$ if and only if $\tau(f(a_1, \dots, a_n)) \le \tau(f(b_1, \dots, b_n))$ for every continuous convex function $f : \mathbb{R}^n \to \mathbb{R}$.*

We end this section with the following elementary fact about diffuse (scalar) measures, i.e. measures without atoms (recall that x is an atom of a measure μ if $\mu(\{x\}) > 0$).

Lemma 2.4. *Let $K \subset \mathbb{R}^n$ be compact and let μ be a regular diffuse Borel probability measure on K. Then for every $\alpha \in (0, 1)$ there exists a measurable set $S \subset K$ such that $\mu(S) = \alpha$.*

3. The local form of doubly stochastic maps

A linear map $\Phi : \mathcal{M} \to \mathcal{M}$ is said to be **doubly stochastic** [12] if it is unital, positive, and trace preserving. We denote the set of all doubly stochastic maps on $\mathcal{M}$ by $\mathrm{DS}(\mathcal{M})$. Doubly stochastic maps play an important role in the theory of majorization between self-adjoint operators (see for instance [1, 2, 12, 13]); thus, the study of their structure appears as a natural topic here.

In what follows we introduce some terminology, we state Theorem 3.1, Proposition 3.2, and Lemma 3.5 and then we use them to prove Theorem 3.6. The proofs of these results will be presented at the end of the paper, in section 6. Although technical, they seem to have some interest on their own.

Let $\mathcal{A} \subseteq \mathcal{M}$ be an abelian C^*-subalgebra, and let $E_{\mathcal{A}}$ and $\mu_{\mathcal{A}}$ denote the spectral measure and the spectral distribution of $\mathcal{A}$ as defined in section 2.1. If $x \in \Gamma(\mathcal{A})$ is such that $E_{\mathcal{A}}(\{x\}) \neq 0$, we say that x is an atom for $E_{\mathcal{A}}$. The set of atoms of $E_{\mathcal{A}}$ is denoted $\mathrm{At}(E_{\mathcal{A}})$. Since $\mu_{\mathcal{A}} = \tau \circ E_{\mathcal{A}}$, the faithfulness of the trace implies that $\mathrm{At}(\mu_{\mathcal{A}}) = \mathrm{At}(E_{\mathcal{A}})$. We say that $\mathcal{A}$ is **diffuse** if $\mathrm{At}(E_{\mathcal{A}}) = \emptyset$. The following theorem states that spectral measures of a separable $\mathcal{A}$ can be refined in a coherent way.

Theorem 3.1. *Let $\mathcal{A} \subseteq \mathcal{M}$ be a separable abelian C^*-subalgebra. Then there exists $a \in \mathcal{M}_{sa}$ such that $C^*(\mathcal{A}, a)$ is abelian and diffuse.*

Since the atoms of $E_{\mathcal{A}}$ are in correspondence with the set of minimal projections of $L^\infty(\mathcal{A})$, Theorem 3.1 provides a way to embed $\mathcal{A}$ into a separable C^*-subalgebra $\tilde{\mathcal{A}} = C^*(\mathcal{A}, a)$ such that $L^\infty(\tilde{\mathcal{A}})$ has no minimal projections (see Remark 6.3 for further discussion).

Proposition 3.2. *Let $\mathcal{B} \subset \mathcal{M}$ be a separable, diffuse, and abelian C^*-subalgebra. Then there exists an unbounded set $\mathbb{M} \subseteq \mathbb{N}$ such that for every $m \in \mathbb{M}$ there exist $k = k(m)$ partitions of the unity $\{q_i^{t,m}\}_{i=1}^m \subseteq \mathcal{B}' \cap \mathcal{M}$, $1 \le t \le k$, with $\tau(q_i^{t,m}) = 1/m$ $(1 \le i \le m,\ 1 \le t \le k)$, and such that for each $b \in \mathcal{B}$, if we let $\beta_i^{t,m} = m\, \tau(b\, q_i^{t,m})$, then*

$$\lim_{m\to\infty} \left\| \, b - \frac{1}{k} \sum_{t=1}^{k} \left(\sum_{i=1}^{m} \beta_i^{t,m}\, q_i^{t,m} \right) \right\| = 0. \tag{2}$$

Remark 3.3. For fixed m and partitions of the unity $\{q_i^t\}_{i=1}^m$ $1 \le t \le k$, the linear map

$$b \mapsto \frac{1}{k} \sum_{t=1}^{k} \left(\sum_{i=1}^{m} m\, \tau(b\, q_i^t)\, q_i^t \right)$$

is a contraction with respect to the operator norm.

We denote by $\mathcal{D}(\mathcal{M})$ the convex semigroup $\mathcal{D}(\mathcal{M}) = \operatorname{conv}\{Ad\, u : \, u \in \mathcal{U}(\mathcal{M})\}$.

Lemma 3.4. *Let $\{p_i\}_{i=1}^m$, $\{q_i\}_{i=1}^m \subseteq \mathcal{M}$ be partitions of the unity such that $\tau(p_i) = \tau(q_i) = \frac{1}{m}$, and let $T \in DS(\mathcal{M})$. Then there exists $\rho \in \mathcal{D}(\mathcal{M})$ such that if $\beta_1, \ldots, \beta_m \in \mathbb{R}$ and $\alpha_i = m \sum_{j=1}^m \beta_j \tau(T(q_j)\, p_i)$ for $1 \le i \le m$, we have*

$$\sum_{i=1}^m \alpha_i p_i = \rho\left(\sum_{i=1}^m \beta_i\, q_i\right). \tag{3}$$

Proof. Let $\gamma_{i,j} = m\, \tau(T(q_j)\, p_i) \ge 0$; it is then straightforward to verify that $(\gamma_{i,j}) \in \mathbb{R}^{m\times m}$ is a doubly stochastic matrix and, moreover, that $\alpha_i = \sum_{j=1}^m \gamma_{i,j}\, \beta_j$ for every $i = 1, \ldots, m$. By Birkhoff's theorem the doubly stochastic matrix $(\gamma_{i,j})$ can be written as a convex combination of permutation matrices, i.e. $(\gamma_{i,j}) = \sum_{\sigma\in\mathbb{S}_m} \eta_\sigma P_\sigma$, where $\eta_\sigma \ge 0$, $\sum_{\sigma\in\mathbb{S}_m} \eta_\sigma = 1$ and P_σ is the $m \times m$ permutation matrix induced by $\sigma \in \mathbb{S}_m$. Then we have

$$\alpha_i = \sum_{j=1}^m \gamma_{i,j}\, \beta_j = \sum_{\sigma\in\mathbb{S}_m} \eta_\sigma\, \beta_{\sigma(i)} \quad 1 \le i \le m. \tag{4}$$

The fact that $\mathcal{M}$ is a II_1 factor and that the elements of the partitions $\{p_i\}_i$, $\{q_i\}_i$ have the same trace guarantees the existence of unitaries u_σ such that $u_\sigma\, q_{\sigma(i)}\, (u_\sigma)^* = p_i$, $1 \le i \le m$, for every $\sigma \in \mathbb{S}_m$. Indeed, if $\sigma \in \mathbb{S}_m$, the equalities, $\tau(q_{\sigma(i)}) = \tau(p_i)$ imply that there exist partial isometries $v_{i,\sigma} \in \mathcal{M}$ such that $v_{i,\sigma} v_{i,\sigma}^* = p_i$ and $v_{i,\sigma}^* v_{i,\sigma} = q_{\sigma(i)}$ for $i = 1, \ldots, m$. Then $u_\sigma = \sum_{i=1}^m v_{i,\sigma} \in \mathcal{M}$ are the required unitaries. Using equation (4), and letting $\rho(\cdot) = \sum_{\sigma\in\mathbb{S}_m} \eta_\sigma\, u_\sigma\, (\cdot)\, u_\sigma^*$,

$$\begin{aligned}
\sum_{i=1}^m \alpha_i\, p_i &= \sum_{i=1}^m \left(\sum_{\sigma\in\mathbb{S}_m} \eta_\sigma\, \beta_{\sigma(i)}\right) p_i = \sum_{\sigma\in\mathbb{S}_m} \eta_\sigma \left(\sum_{i=1}^m \beta_{\sigma(i)}\, u_\sigma\, q_{\sigma(i)}\, u_\sigma^*\right) \\
&= \sum_{\sigma\in\mathbb{S}_m} \eta_\sigma\;\, u_\sigma \left(\sum_{i=1}^m \beta_i\, q_i\right) u_\sigma^* = \rho\left(\sum_{i=1}^m \beta_i\, q_i\right). \quad \square
\end{aligned}$$

Lemma 3.5. *Let $\mathcal{B} \subset \mathcal{M}$ be a separable C^*-subalgebra, and let $\{p_i\}_{i=1}^m \subseteq \mathcal{B}' \cap \mathcal{M}$ be a partition of the unity. Then there exists a sequence $\{\rho_i\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathcal{M})$ such that for every $b \in \mathcal{B}$, if we let $\beta_i(b) = \tau(b\, p_i)/\tau(p_i)$, then*

$$\lim_{j\to\infty} \left\| \rho_j(b) - \sum_{i=1}^m \beta_i(b) p_i \right\| = 0.$$

Theorem 3.6. *Let $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{M}$ be separable abelian C^*-subalgebras and let $T \in DS(\mathcal{M})$. Let $\mathcal{S}$ be the operator subsystem of $\mathcal{B}$ given by $\mathcal{S} = T^{-1}(\mathcal{A}) \cap \mathcal{B}$. Then there exists a sequence $(\rho_r)_{r\in\mathbb{N}} \subseteq \mathcal{D}(\mathcal{M})$ such that $\lim_{r\to\infty} \|T(b) - \rho_r(b)\| = 0$ for every $b \in \mathcal{S}$.*

Proof. First, note that we just have to prove the theorem for separable diffuse abelian C^*-subalgebras of $\mathcal{M}$; indeed, assume it holds for such algebras and let $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{M}$ be arbitrary separable abelian C^*-subalgebras. Then, by Theorem 3.1 there exist separable diffuse abelian subalgebras $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ of $\mathcal{M}$ such that $\mathcal{A} \subseteq \tilde{\mathcal{A}}$ and $\mathcal{B} \subseteq \tilde{\mathcal{B}}$. Thus we get a sequence $\{\rho_r\}_{r\in\mathbb{N}} \subseteq \mathcal{D}$ such that $\lim_{r\to\infty} \|T(b) - \rho_r(b)\| = 0$, for every $b \in T^{-1}(\mathcal{A}) \cap \mathcal{B} \subseteq T^{-1}(\tilde{\mathcal{A}}) \cap \tilde{\mathcal{B}}$. So we assume that $\mathcal{A}$ and $\mathcal{B}$ are diffuse.

By Proposition 3.2, there exists an unbounded set $\mathbb{M} \subseteq \mathbb{N}$ and, for each $m \in \mathbb{M}$, $k(m)$ partitions of the unity $\{q_i^{j,m}\}_{i=1}^m \subseteq \mathcal{B}' \cap \mathcal{M}$ and $\{p_i^{j,m}\}_{i=1}^m \subseteq \mathcal{A}' \cap \mathcal{M}$ (in order to simplify

the notation we avoid the supra-index m and write q_i^j, p_i^j), $1 \le j \le k$, such that for every $b \in T(\mathcal{A})^{-1} \cap \mathcal{B}$ and every $r \in \mathbb{N}$, there exists $m_0(r,b) \in \mathbb{M}$ such that if $m \ge m_0$ we have

$$\left\| b - \frac{1}{k} \sum_{j=1}^{k} \left(\sum_{i=1}^{m} \beta_i^j q_i^j \right) \right\| < \frac{1}{r} \tag{5}$$

and

$$\left\| T(b) - \frac{1}{k} \sum_{j=1}^{k} \left(\sum_{i=1}^{m} \alpha_i^j p_i^j \right) \right\| < \frac{1}{r} \tag{6}$$

where $\beta_i^j = m\,\tau(b\,q_i^j)$, $\alpha_i^j = m\,\tau(T(b)\,p_i^j)$, $\tau(p_i^j) = \tau(q_i^j) = 1/m$, (from the construction of such partitions it is evident that we can assume that both have the same unbounded set $\mathbb{M}$ and the same $k(m)$ for every $m \in \mathbb{M}$). Fix $b \in \mathcal{B}$. Since $\|T\| = 1$, it follows from equation (5) that

$$\left\| T(b) - \frac{1}{k} \sum_{j=1}^{k} \left(\sum_{i=1}^{m} \beta_i^j \, T(q_i^j) \right) \right\| \le \frac{1}{r}. \tag{7}$$

Applying to (7) the fact that the linear map in Remark 3.3 is linear and contractive (with $\{p_i^j\}_i$ as the partitions of the unity), we get

$$\left\| \frac{1}{k} \sum_{j=1}^{k} \sum_{i=1}^{m} \alpha_i^j \, p_i^j - \frac{1}{k^2} \sum_{j=1}^{k} \left(\sum_{t=1}^{k} \sum_{i=1}^{m} \alpha_i^{j,\,t} \, p_i^t \right) \right\| \le \frac{1}{r}, \tag{8}$$

where $\alpha_i^{j,\,t} = m \sum_{l=1}^m \beta_l^j \, \tau(T(q_l^j) p_i^t)$, and α_i^j as defined above. By Lemma 3.4 there exists $\rho_{j,\,t}^m \in \mathcal{D}(\mathcal{M})$ such that

$$\sum_{i=1}^{m} \alpha_i^{j,\,t} p_i^t = \rho_{j,\,t}^m \left(\sum_{l=1}^{m} \beta_l^j q_l^j \right), \qquad 1 \le j, t \le k. \tag{9}$$

Using (6), (8), and (9) we get

$$\left\| T(b) - \frac{1}{k^2} \sum_{j=1}^{k} \sum_{t=1}^{k} \rho_{j,\,t}^m \left(\sum_{l=1}^{m} \beta_l^j \, q_l^j \right) \right\| \le \frac{2}{r}, \tag{10}$$

By Lemma 3.5 there exist sequences $(\tilde{\rho}_n^j)_{n \in \mathbb{N}} \subseteq \mathcal{D}(\mathcal{M})$, $1 \le j \le k$, independent of b, such that for every $r \in \mathbb{N}$ there exists $n_0 = n_0(r,b)$ such that if $n \ge n_0$ then

$$\left\| \sum_{l=1}^{m} \beta_l^j \, q_l^j - \tilde{\rho}_n^j(b) \right\| \le \frac{1}{r}, \quad 1 \le j \le k. \tag{11}$$

From (10) and (11), together with the fact that each $\rho \in \mathcal{D}(\mathcal{M})$ is contractive we get, for every $n \ge n_0(r,b)$

$$\left\| T(b) - \frac{1}{k^2} \sum_{j=1}^{k} \sum_{t=1}^{k} \rho_{j,\,t}^m (\tilde{\rho}_n^j(b)) \right\| \le \frac{3}{r}, \tag{12}$$

Consider a dense countable subset $\{b_1, b_2, \ldots\}$ of $\mathcal{B}$. Now define $n(r)$, $m(r)$ as $n(r) = \max\{n_0(r,b_1), \ldots, n_0(r,b_r)\}$ and $m(r) = \max\{m_0(r,b_1), \ldots, m_0(r,b_r)\}$ and let $\rho_r := \frac{1}{k^2} \sum_{j=1}^k \sum_{t=1}^k \rho_{j,\,t}^{m(r)} \circ \tilde{\rho}_{n(r)}^j \in \mathcal{D}(\mathcal{M})$, where $k = k(m(r))$. Then, from the previous calculations, we see that $\|T(b_j) - \rho_r(b_j)\| < 3/r$ whenever $1 \le j \le r$. Let $b \in \mathcal{B}$, and $\epsilon > 0$. Then there exists $l \in \mathbb{N}$ such that $\|b - b_l\| < \epsilon/3$. If $r > \max\{l, 9/\epsilon\}$, then $\|T(b_l) - \rho_r(b_l)\| < \epsilon/3$, and so $\|T(b) - \rho_r(b)\| \le \epsilon$. $\square$

Corollary 3.7. *Let $T \in DS(\mathcal{M})$ and let $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n \subseteq \mathcal{M}_{sa}$ be abelian families such that $T(b_i) = a_i$ for $1 \le i \le n$. Then there exists a sequence $(\rho_r)_{r\in\mathbb{N}} \subseteq \mathcal{D}$ such that for $1 \le i \le n$ $\lim_{r\to\infty} \|a_i - \rho_r(b_i)\| = 0$.*

Proof. Consider $\mathcal{A} = C^*(a_1, \dots, a_n)$ and $\mathcal{B} = C^*(b_1, \dots, b_n)$, which are separable abelian C^*-subalgebras of $\mathcal{M}$. Applying Theorem 3.6 to this algebras we get a sequence $(\rho_r)_{r\in\mathbb{N}} \subseteq \mathcal{D}$ such that $\lim_{r\to\infty} \|T(b) - \rho_r(b)\| = 0$ for every $b \in T^{-1}(\mathcal{A}) \cap \mathcal{B}$. By our choice, $b_i \in T^{-1}(\mathcal{A}) \cap \mathcal{B}$ and so $\|T(b_i) - \rho_r(b_i)\| = \|a_i - \rho_r(b_i)\| \xrightarrow{r} 0$. □

4. Doubly stochastic kernels and joint majorization

We begin by introducing doubly stochastic kernels, which are a natural generalization of doubly stochastic matrices. We shall use them to define joint majorization in analogy with [19].

Definition 4.1. *Let (X, μ_X), (Y, μ_Y) be two probability spaces. A positive unital linear map $\nu : L^\infty(Y, \mu_Y) \to L^\infty(X, \mu_X)$ is said to be a* **doubly stochastic kernel** *if $\int_X \nu(1_\Delta)\, d\mu_X = \mu_Y(\Delta)$, for every μ_Y-measurable set $\Delta \subseteq Y$.*

Doubly stochastic kernels between probability spaces are norm continuous and normal.

Example 4.2. Let X and Y be compact spaces and let μ_X and μ_Y be regular Borel probability measures in X and Y respectively. Consider $D \in L^1(\mu_X \times \mu_Y)$ and let $\nu(f)(x) = \int_Y D(x, y)\, f(y)\, d\mu_Y(y)$. Then $\nu : L^\infty(X, \mu_X) \to L^\infty(Y, \mu_Y)$ is a doubly stochastic kernel if and only if $D(x, y) \ge 0$ $(\mu_X \times \mu_Y)$-a.e. and $\int_X D(x, y)\, d\mu_X(x) = 1$ μ_Y-a.e, $\int_Y D(x, y)\, d\mu_Y(y) = 1$ μ_X-a.e. In particular, if $\mu_X = \mu_Y$ is a measure with finite support $\{x_i\}_{i=1}^m$ and such that $\mu_X(\{x_i\}) = \frac{1}{m}$ for $1 \le i \le m$ then D is a doubly stochastic kernel if and only if the matrix $(D(x_i, x_j))_{i,\,j}$ is an $m \times m$ doubly stochastic matrix.

Proposition 4.3. *Let $\bar{a} = (a_i)_{i=1}^n$, $\bar{b} = (b_i)_{i=1}^n \subseteq \mathcal{M}_{sa}$ be abelian families. Then the following statements are equivalent:*

1. *There exists $T \in DS(\mathcal{M})$ such that $T(b_i) = a_i$, $1 \le i \le n$.*
2. *There exists a doubly stochastic kernel $\nu : L^\infty(\sigma(\bar{b}), \mu_{\bar{b}}) \to L^\infty(\sigma(\bar{a}), \mu_{\bar{a}})$ such that $\nu(\pi_i) = \pi_i$, $1 \le i \le n$.*

Proof. Assume that $T(b_i) = a_i$, $1 \le i \le n$, with $T \in DS(\mathcal{M})$. Let $\mathcal{A} = C^*(a_1, \dots, a_n)$, $\mathcal{B} = C^*(b_1, \dots, b_n)$. As $\mathcal{M}$ is a finite von Neumann algebra, there exists a conditional expectation $\mathcal{P}_\mathcal{A} : M \to L^\infty(\mathcal{A})$ that commutes with τ. Then $\nu = \Lambda_{\bar{a}}^{-1} \circ \mathcal{P}_\mathcal{A} \circ T \circ \Lambda_{\bar{b}}$ is the desired doubly stochastic kernel. Conversely, let us assume the existence of ν as in *2*. Let $\mathcal{P}_\mathcal{B} : \mathcal{M} \to L^\infty(\mathcal{B})$ be the conditional expectation onto $L^\infty(\mathcal{B})$ that commutes with τ. Then define $T = \Lambda_{\bar{a}} \circ \nu \circ \Lambda_{\bar{b}}^{-1} \circ \mathcal{P}_\mathcal{B} \in DS(\mathcal{M})$. Clearly $T(b_i) = a_i$, $1 \le i \le n$. □

Definition 4.4. *Let $\bar{a} = (a_i)_{i=1}^n$, $\bar{b} = (b_i)_{i=1}^n$ be two abelian families in $\mathcal{M}_{sa}$. We say that $\bar{a}$ is* **jointly majorized** *by $\bar{b}$ (and we write $\bar{a} \prec \bar{b}$) if there exists a doubly stochastic kernel $\nu : L^\infty(\sigma(\bar{b}), \mu_{\bar{b}}) \to L^\infty(\sigma(\bar{a}), \mu_{\bar{a}})$ such that $\nu(\pi_i) = \pi_i$, $1 \le i \le n$.*

If $(x_1, \dots, x_n)$ is a finite family in $\mathcal{M}$, let $\mathcal{U}_\mathcal{M}(x_1, \dots, x_n)$ denote the **joint unitary orbit** of the family with respect to the unitary group $\mathcal{U}_\mathcal{M}$ of $\mathcal{M}$, i.e.

$$\mathcal{U}_\mathcal{M}(x_1, \dots, x_n) = \{(u^* x_1 u, \dots, u^* x_n u) : \, u \in \mathcal{U}_\mathcal{M}\}.$$

We shall also consider the convex hull of the unitary orbit of a family $(x_i)_{i=1}^n$,

$$\operatorname{conv}(\mathcal{U}_\mathcal{M}(x_i)_{i=1}^n) = \{(\rho(x_i))_{i=1}^n, \ \rho \in \mathcal{D}\}.$$

We denote by $\overline{\operatorname{conv}}(\mathcal{U}_\mathcal{M}(x_i)_{i=1}^n)$, $\overline{\operatorname{conv}}^w(\mathcal{U}_\mathcal{M}(x_i)_{i=1}^n)$ and $\overline{\operatorname{conv}}^1(\mathcal{U}_\mathcal{M}(x_i)_{i=1}^n)$ the respective closures in the coordinate-wise norm topology, coordinate-wise weak operator topology, and coordinate-wise L^1 topology.

Theorem 4.5. *Let $\bar{a} = (a_i)_{i=1}^n$, $\bar{b} = (b_i)_{i=1}^n$ be abelian families in $\mathcal{M}_{sa}$. Then the following statements are equivalent:*

(1) *$\bar{a}$ is jointly majorized by $\bar{b}$.*
(2) *$\bar{a} \in \overline{\text{conv}}(\mathcal{U}_{\mathcal{M}}(\bar{b}))$.*
(3) *$\bar{a} \in \overline{\text{conv}}^{\,1}(\mathcal{U}_{\mathcal{M}}(\bar{b}))$.*
(4) *$\bar{a} \in \overline{\text{conv}}^{\,w}(\mathcal{U}_{\mathcal{M}}(\bar{b}))$.*
(5) *$\mu_{\bar{a}} \prec \mu_{\bar{b}}$.*
(6) *There exists a completely positive map $T \in DS(\mathcal{M})$ such that $a_i = T(b_i)$, $1 \le i \le n$.*
(7) *There exists $T \in DS(\mathcal{M})$ such that $a_i = T(b_i)$, $1 \le i \le n$.*
(8) *$\tau(f(a_1, \ldots, a_n)) \le \tau(f(b_1, \ldots, b_n))$ for every continuous convex function $f : \mathbb{R}^n \to \mathbb{R}$.*

Remark 4.6. Let $x \in \mathcal{M}$ be a normal operator. Recall (see the last paragraph of section 2.1) that there is a natural way to identify the usual spectral measure of x with that of the abelian pair $(\text{Re}(x), \text{Im}(x))$. If $T \in DS(\mathcal{M})$, then since T is positive $T(x) = y$ if and only if $T(\text{Re}(x)) = \text{Re}(y)$ and $T(\text{Im}(x)) = \text{Im}(y)$. From these facts and Theorem 4.5, we see that if x, $y \in \mathcal{M}$ are normal operators then $x \prec y$ in the sense of [13] if and only if $(\text{Re}(x), \text{Im}(x)) \prec (\text{Re}(y), \text{Im}(y))$ in the sense of Definition 4.4.

Let $\mathcal{P}_{\mathcal{N}}$ denote the trace preserving conditional expectation onto the abelian von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$. Using Theorem 4.5 we can then obtain a generalization of Theorem 7.2 in [8].

Corollary 4.7. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an abelian von Neumann subalgebra and let $(b_i)_{i=1}^n \subseteq \mathcal{M}_{sa}$ be an abelian family. Then $(\mathcal{P}_{\mathcal{N}}(b_i))_{i=1}^n \prec (b_i)_{i=1}^n$.*

In the remainder of the section we prove the implications needed to prove Theorem 4.5. The single variable case of the following lemma can be found in [13].

Lemma 4.8. *Let $\bar{a} = (a_i)_{i=1}^n$, $\bar{b} = (b_i)_{i=1}^n \subseteq \mathcal{M}_{sa}$ be abelian families. If $\bar{a} \in \overline{\text{conv}}^{\,w}(\mathcal{U}_{\mathcal{M}}(\bar{b}))$ then there exists a completely positive $T \in DS(\mathcal{M})$ such that $a_i = T(b_i)$, $1 \le i \le n$.*

Proof. Let $\{(b_1^j, \ldots, b_n^j)\}_{j \in J} \subseteq \text{conv}(\mathcal{U}_{\mathcal{M}}(b_1, \ldots, b_n))$ such that $b_i^j \xrightarrow[j]{\text{weakly}} a_i$, $1 \le i \le n$. Then there exists a sequence $(\rho_j)_{j \in J} \subseteq \mathcal{D}$ such that $(b_1^j, \ldots, b_n^j) = (\rho_j(b_1), \ldots, \rho_j(b_n))$, for every $j \in J$. Note that ρ_j is a completely positive doubly stochastic map and the net $\{\rho_j\}_{j \in J}$ is norm bounded. Therefore this net has an accumulation point in the BW topology [7], i.e. there exists a subnet (which we still call $\{\rho_j\}_{j \in J}$) and a completely positive map $T : \mathcal{M} \to \mathcal{M}$ such that $\rho_j(x) \xrightarrow[j]{\text{weakly}} T(x)$ if $x \in \mathcal{M}$. By normality of the trace, T is trace preserving, positive and unital. Since $\rho_j(b_i) = b_i^j \xrightarrow[j]{\text{weakly}} a_i$, we have $T(b_i) = a_i$, $1 \le i \le n$. ☐

Lemma 4.9. *Let $\bar{a} = (a_i)_{i=1}^n$, $\bar{b} = (b_i)_{i=1}^n \subseteq \mathcal{M}_{sa}$ be abelian families. If $\bar{a} \prec \bar{b}$, then $\mu_{\bar{a}} \prec \mu_{\bar{b}}$.*

Proof. By hypothesis, $\bar{a} \prec \bar{b}$; that is, there exists a doubly stochastic kernel $\nu : L^\infty(\sigma(\bar{b}), \mu_{\bar{b}}) \to L^\infty(\sigma(\bar{a}), \mu_{\bar{a}})$ such that $\nu(\pi_i) = \pi_i$, $1 \le i \le n$. Let $\nu_1, \ldots, \nu_m \in M_+^\sim(\mathbb{R}^n)$ with $\sum_{j=1}^m \nu_j = \mu_{\bar{a}}$. Define measures ν'_j by $\nu'_j(\Delta) = \nu_j(\nu(1_\Delta))$. By continuity of ν, $\nu'_j(f) = \nu_j(\nu(f))$ for every $f \in L^\infty(\sigma(\bar{b}), \mu_{\bar{b}})$. So $\nu'_j(\pi_i) = \nu_j(\nu(\pi_i)) = \nu_j(\pi_i)$, $1 \le i \le n$ and $1 \le j \le m$, and similarly $\nu_j(1) = \nu'_j(1)$, so that $\nu_j \sim \nu'_j$, for $1 \le j \le m$. Finally, $\sum_{j=1}^m \nu'_j(\Delta) = \sum_{j=1}^m \nu_j(\nu(1_\Delta)) = \mu_{\bar{a}}(\nu(1_\Delta)) = \mu_{\bar{b}}(\Delta)$. Therefore $\sum_{j=1}^m \nu'_j = \mu_{\bar{b}}$. We conclude that $\mu_{\bar{a}} \prec \mu_{\bar{b}}$. ☐

Lemma 4.10. *Let $\bar{a} = (a_i)_{i=1}^n$, $\bar{b} = (b_i)_{i=1}^n \subset \mathcal{M}_{sa}$ be abelian families. If $\mu_{\bar{a}} \prec \mu_{\bar{b}}$, then there exists $T \in DS(\mathcal{M})$ such that $T(b_i) = a_i$, $1 \le i \le n$.*

Proof. By compactness, we can consider partitions $\{\Delta_j^r\}_{j=1}^{m(r)}$ of $\sigma(\bar a)$ with $\operatorname{diam}(\Delta_j^r) < 1/r$ for every $1 \le j \le m$. Fix points $x_1^r, \ldots, x_{m(r)}^r$ with $x_j^r \in \Delta_j^r$ and define measures μ_j^r by $\mu_j^r(\cdot\,) = \mu_{\bar a}(\cdot \cap \Delta_j^r)$. Then clearly $\sum_j \mu_j^r = \mu_{\bar a}$. As $\mu_{\bar a} \prec \mu_{\bar b}$ by hypothesis, there exist measures ν_j^r with $\nu_j^r \sim \mu_j^r$ and $\sum_j \nu_j^r = \mu_{\bar b}$. Let g_j^r be the Radon-Nikodym derivatives $g_j^r = d\nu_j^r/d\mu_{\bar b}$. Note that $\sum_j g_j^r = 1$ ($\mu_{\bar b}$ – a.e.). Define a function $D_r : \sigma(\bar a) \times \sigma(\bar b) \to \mathbb{R}$ by

$$D_r(s,t) = \sum_{j=1}^{m(r)} \frac{g_j^r(t)}{\mu_a(\Delta_j^r)}\, 1_{\Delta_j^r}(s).$$

We will use the kernels D_r to approximate T. Let us define $\nu_r : L^\infty(\sigma(\bar b), \mu_{\bar b}) \to L^\infty(\sigma(\bar a), \mu_{\bar a})$ by

$$\nu_r(b)(s) = \int_{\sigma(\bar b)} b(t)\, D_r(s,t)\, d\mu_{\bar b}(t).$$

The map ν_r can be seen to be doubly stochastic using the equivalence $\mu_j^r \sim \nu_j^r$. By Proposition 4.3 there is an associated sequence $\{T_r\}_r \subset DS(\mathcal{M})$ such that $T_r(b_i) = \int_{\sigma(\bar a)} \nu_r(\pi_i)\, dE_{\bar a} \in L^\infty(\mathcal{A})$, $1 \le i \le n$. The bounded net $\{T_r\}_{r\in\mathbb{N}}$ has a subnet $\{T_k\}_{k\in K}$ that converges to a cluster point $T \in DS(\mathcal{M})$ in the BW topology. Since this subnet is bounded, $T(b_i) = w\text{-}\lim_{k\in K} T_k(b_i) \in L^\infty(\mathcal{A})$. We claim that $T(b_i) = a_i$, $1 \le i \le n$. To see this, since the net $\{T_k(b_i)\}_{k\in K}$ is bounded, we just have to prove that

$$\lim_k \tau(x\, T_k(b_i)) = \tau(x\, a_i), \quad 1 \le i \le n, \quad \forall x \in \mathcal{A}.$$

Equivalently, we have to show that for every continuous function $f \in C(\sigma(\bar a))$ and every $i = 1, \ldots, n$,

$$\lim_k \int_{\sigma(\bar a)} f(s) \left(\int_{\sigma(\bar b)} D_k(s,t)\, \pi_i(t)\, d\mu_{\bar b}(t) \right) d\mu_{\bar a}(s) = \int_{\sigma(\bar a)} f(s)\, \pi_i(s)\, d\mu_{\bar a}(s).$$

This can be seen by a standard approximation argument, using the uniform continuity of f, the fact that the diameters of Δ_j^r tend to 0 as r increases, and the equivalence $\mu_j^r \sim \nu_j^r$. ☐

Proof of Theorem 4.5. Proposition 4.3 shows the equivalence (7)⇔(1) and Corollary 3.7 is (7)⇒(2). The implication (2)⇒ (3)⇒(4) is trivial. Lemma 4.8 shows that (4)⇒(6), and it is clear that (6)⇒(7). Lemmas 4.9, 4.10 and Proposition 4.3 prove the equivalence (5)⇔(1). So we have that (1)-(7) are equivalent. Finally, Corollary 2.3 shows that (5)⇔(8). ☐

5. Joint unitary orbits of abelian families in $\mathcal{M}_{sa}$

Given families $\bar a = (a_i)_{i=1}^n$, $\bar b = (b_i)_{i=1}^n \subseteq \mathcal{M}$, we say that $\bar a$ and $\bar b$ are **jointly approximately unitarily equivalent in** $\mathcal{M}$ if $\bar a \in \overline{\mathcal{U}_{\mathcal{M}}(\bar b)}$, that is if there exists a sequence of unitary operators $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that $\lim_{n\to\infty} \|u_n b_i u_n^* - a_i\| = 0$ for every $i = 1\ldots, n$. It is clear that this is an equivalence relation. Moreover, if $\bar a$ and $\bar b$ are jointly approximately unitarily equivalent in $\mathcal{M}$ then $\bar a$ is an abelian family if and only if $\bar b$ is. In [8] a characterization of this equivalence relation between selfadjoint operators is obtained, in terms of the spectral distributions. The following results exhibits a list of characterizations of this relation for abelian families in $\mathcal{M}_{sa}$.

Theorem 5.1. *Let $\bar a = (a_i)_{i=1}^n$ and $\bar b = (b_i)_{i=1}^n \subset \mathcal{M}_{sa}$ be abelian families. Then the following statements are equivalent:*

1. *$\bar a$ and $\bar b$ are jointly approximately unitary equivalent in $\mathcal{M}$.*
2. *$\bar a \prec \bar b$ and $\bar b \prec \bar a$*
3. *$\mu_{\bar a} = \mu_{\bar b}$*
4. *$\tau(f(a_1, \ldots, a_n)) = \tau(f(b_1, \ldots, b_n))$ for every continuous convex function $f : \mathbb{R}^n \to \mathbb{R}$.*
5. *$\tau(f(a_1, \ldots, a_n)) = \tau(f(b_1, \ldots, b_n))$ for every continuous function $f : \mathbb{R}^n \to \mathbb{R}$.*

Proof. By Theorem 4.5 we have (1)⇒(2) and (2)⇔(4). Moreover, (4) is equivalent to $\mu_{\bar a}(f) = \mu_{\bar b}(f)$ for every convex function f. Then $\mu_{\bar a}(f) = \mu_{\bar b}(f)$ for every continuous function f [3, Proposition I.1.1], and this in turn implies that $\mu_{\bar a} = \mu_{\bar b}$. Therefore, (4)⇒(5)⇒(3). Again, by Theorem 4.5 (3)⇒(2) and so (2)-(5) are equivalent. Finally, we prove that (3)⇒(1). If we assume that $\mu_{\bar a} = \mu_{\bar b}$ then, $\sigma(\bar a) = \text{supp } \mu_{\bar a} = \text{supp } \mu_{\bar b} = \sigma(\bar b)$ and for every Borel set Δ in $\sigma(\bar a)$ we have

$$\tau(E_{\bar a}(\Delta)) = \mu_{\bar a}(1_\Delta) = \mu_{\bar b}(1_\Delta) = \tau(E_{\bar b}(\Delta)). \tag{13}$$

Let $\epsilon > 0$. By compactness, choose $B_1, \ldots, B_m$ to be a finite disjoint covering of $\sigma(\bar a) = \sigma(\bar b)$ such that there are points $x_j \in B_j$ with the property that $|\pi_i(\lambda) - \pi_i(x_j)| < \epsilon/2$ for every $\lambda \in B_j$, $1 \le i \le n$, $1 \le j \le m$. Then we get, using the Spectral Theorem,

$$\left\| a_i - \sum_{j=1}^m \pi_i(x_j) E_{\bar a}(B_j) \right\| < \frac{\epsilon}{2}, \quad \left\| b_i - \sum_{j=1}^m \pi_i(x_j) E_{\bar b}(B_j) \right\| < \frac{\epsilon}{2}$$

for $i = 1, \ldots, n$. From equation (13) we get that $\tau(E_{\bar a}(B_j)) = \tau(E_{\bar b}(B_j))$ for every $j = 1, \ldots, m$. As in the proof of Lemma 3.4, we get a unitary $w_\epsilon \in \mathcal{U}(\mathcal{M})$ such that $w_\epsilon^* E_{\bar b}(B_j) w_\epsilon = E_{\bar a}(B_j)$ for every j. Then

$$w_\epsilon^* \left(\sum_{j=1}^m \pi_i(x_j) E_{\bar b}(B_j) \right) w_\epsilon = \sum_{j=1}^m \pi_i(x_j) E_{\bar a}(B_j).$$

Finally, for every i we have

$$\| w_\epsilon^* b_i w_\epsilon - a_i \| \le \left\| w_\epsilon^* \left(b_i - \sum_{j=1}^m \pi_i(x_j) E_{\bar b}(B_j) \right) w_\epsilon \right\| + \frac{\epsilon}{2} < \epsilon. \quad \square$$

Corollary 5.2. *Let Θ be an automorphism of $\mathcal{M}$. Then $\Theta|_{\mathcal{A}}$ is approximately inner for each separable abelian C^* subalgebra $\mathcal{A} \subset \mathcal{M}$.*

Proof. The uniqueness of the trace guarantees that Θ is trace-preserving. Being multiplicative, the range of an abelian set will be again abelian. So Θ is a DS map that takes an abelian family in $\mathcal{M}$ into another. Consider a countable dense subset $\{a_i\}$ of $\mathcal{A}$, and use Theorem 5.1 to obtain unitaries u_n for each finite subset $\{a_1, \ldots, a_n\}$. An $\epsilon/3$ argument shows then that the sequence $\{\operatorname{Ad} u_n\}$ approximates Θ in all of $\mathcal{A}$. $\square$

Given $\bar x = (x_i)_{i=1}^n \subseteq \mathcal{M}$ we denote by $\overline{\mathcal{U}_{\mathcal{M}}(\bar x)}^s$ the closure in the coordinate-wise strong operator topology. An immediate consequence of Theorem 5.1 is that the norm closure of the unitary orbit of a selfadjoint abelian family in a II$_1$ factor is strongly closed. This generalizes [8, Theorem 5.4] and [22, Theorem 8.12(1)]:

Corollary 5.3. *Let $\bar a = (a_i)_{i=1}^n \subseteq \mathcal{M}_{sa}$ be an abelian family. Then $\overline{\mathcal{U}_{\mathcal{M}}(\bar a)}^{\|\,\|} = \overline{\mathcal{U}_{\mathcal{M}}(\bar a)}^s$.*

Proof. Let $\bar b = (b_i)_{i=1}^n \in \overline{\mathcal{U}_{\mathcal{M}}(\bar a)}^s$. There exists a net $(b_1^j, \ldots, b_n^j)_{j \in J} \subseteq \mathcal{U}_{\mathcal{M}}(\bar a)$ such that b_i^j converges strongly to b_i for each $i = 1, \ldots, n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then $\tau(f(b_1^j, \ldots, b_n^j)) = \tau(f(a_1, \ldots, a_n))$ for every j. Using [23, Lemma II.4.3] we conclude that $\tau(f(b_1, \ldots, b_n)) = \tau(f(a_1, \ldots, a_n))$. So (5) of Theorem 5.1 implies that $\bar b \in \overline{\mathcal{U}_{\mathcal{M}}(\bar a)}$. The other inclusion is trivial. $\square$

6. Some technical results

In this section we prove the results presented at the beginning of section 3. First, we show that any separable abelian C^*-subalgebra of $\mathcal{M}$ can be embedded into a separable diffuse abelian C^*-subalgebra. Then, we prove some approximation results that hold for separable diffuse abelian C^* subalgebras of $\mathcal{M}$.

6.1. **Refinements of spectral measures.** We begin by recalling some elementary facts about inclusions of abelian C^* algebras. If $\mathcal{A} \subseteq \mathcal{B}$ are unital C^*-algebras, then the function $\Phi : \Gamma(\mathcal{B}) \to \Gamma(\mathcal{A})$ given by $\Phi(\gamma) = \gamma|_{\mathcal{A}}$ is a continuous surjection onto $\Gamma(\mathcal{A})$. If we assume further that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$ are separable and that $E_{\mathcal{A}}$, $E_{\mathcal{B}}$ denote their spectral measures, then $E_{\mathcal{A}} = E_{\mathcal{B}} \circ \Phi^{-1}$ and $\mu_{\mathcal{A}} = \mu_{\mathcal{B}} \circ \Phi^{-1}$.

Note that $\mathrm{At}(\mu_{\mathcal{A}})$ $=\mathrm{At}(E_{\mathcal{A}})$ where $\mathrm{At}(E_{\mathcal{A}})$ is the set of atoms of the spectral measure $E_{\mathcal{A}}$ (see the beginning of section 3). Let $\sum_{x\in \mathrm{At}(E_{\mathcal{A}})} \mu_{\mathcal{A}}(\{x\})$ be the **total atomic mass** of $E_{\mathcal{A}}$. Since $\mu_{\mathcal{A}}$ is finite, the total atomic mass is bounded and thus, the set of atoms is countable set.

Lemma 6.1. *With the notation above, if $x \in At(E_{\mathcal{B}})$ then $\Phi(x) \in At(E_{\mathcal{A}})$, and the total atomic mass of $\mathcal{B}$ is smaller that the total atomic mass of $\mathcal{A}$.*

Proof. Let $x \in \mathrm{At}(E_{\mathcal{B}})$ and note that $0 \neq \mu_{\mathcal{B}}(\{x\}) \leq \mu_{\mathcal{B}}(\Phi^{-1}(\Phi(\{x\}))) = \mu_{\mathcal{A}}(\Phi(\{x\}))$, so $\Phi(x) \in \mathrm{At}(E_{\mathcal{A}}) = \mathrm{At}(\mu_{\mathcal{A}})$. We consider the equivalence relation in $\mathrm{At}(E_{\mathcal{B}})$ induced by Φ, i.e. $x \sim y$ if $\Phi(x) = \Phi(y)$. If $Q \in \mathcal{Q} := \mathrm{At}(E_{\mathcal{B}})/\sim$ is such that $\Phi(x) = x_Q$ for every $x \in Q$, then using that Q is countable we get $\sum_{x\in Q} \mu_{\mathcal{B}}(\{x\}) = \mu_{\mathcal{B}}(Q) \leq \mu_{\mathcal{B}}(\Phi^{-1}(\{x_Q\})) = \mu_{\mathcal{A}}(\{x_Q\})$. Therefore

$$\sum_{x\in \mathrm{At}(E_{\mathcal{B}})} \mu_{\mathcal{B}}(\{x\}) = \sum_{Q\in\mathcal{Q}} \sum_{x\in Q} \mu_{\mathcal{B}}(\{x\}) \leq \sum_{Q\in\mathcal{Q}} \mu_{\mathcal{A}}(x_Q) \leq \sum_{x\in \mathrm{At}(E_{\mathcal{A}})} \mu_{\mathcal{A}}(\{x\}).$$

□

Proposition 6.2. *With the notations above, let $x_0 \in \Gamma(\mathcal{A})$ be an atom of $E_{\mathcal{A}}$ and let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$. Then there exists $a \in \mathcal{A}' \cap \mathcal{M}_{sa}$ with $[\alpha, \beta] \subseteq \sigma(a) \subseteq [\alpha, \beta] \cup \{0\}$, $P_{\overline{R(a)}} = E_{\mathcal{A}}(\{x_0\})$, and such that $E_{\mathcal{B}}$ has no atoms in the fibre $\Phi^{-1}(x_0)$, where $\mathcal{B} = C^*(\mathcal{A}, a) \subset \mathcal{M}$.*

Proof. Let $p = E_{\mathcal{A}}(\{x_0\})$ and consider a masa $\mathcal{A} \subset \mathcal{M}$ such that $p \in \mathcal{A}$. Then $p\mathcal{A}$ is a masa in the II_1 factor $p\mathcal{M}p$, where the trace is $\tau_p = \frac{1}{\tau(p)}\,\tau$. It is well known that there exists a countably generated, non-atomic von Neumann subalgebra $\mathcal{B}$ of $p\mathcal{A}$ such that there is a von Neumann algebra isomorphism $\Phi : L^\infty([0,1], m) \to \mathcal{B}$, with m the Lebesgue measure on $[0,1]$, and with $\tau_p(\Phi(f)) = \int_0^1 f\, dm$. Put $\tilde{a} = \Phi(id)$; it is clear that $\tilde{a}$ has no atoms in its spectrum with the exception of 0, and that $E_{\tilde{a}}(\{0\}) = 1 - p$, $\sigma(a) = [0,1]$. Let $a = (\beta - \alpha)\tilde{a} + \alpha\, p$, so $[\alpha, \beta] \subseteq \sigma(a) \subseteq [\alpha, \beta] \cup \{0\}$, $P_{\overline{R(a)}} = p = E_{\mathcal{A}}(\{x_0\})$. As p is a minimal projection in $L^\infty(\mathcal{A})$, we have $pb = pbp = \lambda_b p$ for every $b \in \mathcal{A}$ and so $ab = apb = \lambda_b pa = bpa = ba$. Thus $a \in \mathcal{A}' \cap \mathcal{M}$.

Let $\mathcal{B} = C^*(\mathcal{A}, a)$ and let $\Phi : \Gamma(\mathcal{B}) \to \Gamma(\mathcal{A})$, $\Psi : \Gamma(\mathcal{B}) \to \Gamma(C^*(a))$ be the continuous surjections induced by the inclusions $\mathcal{A} \subseteq \mathcal{B}$ and $C^*(a) \subseteq \mathcal{B}$. Note that the restriction $\Psi|_{\Phi^{-1}(x_0)}$ is injective. Indeed, let $x, y \in \Phi^{-1}(x_0)$ be such that $\Psi(x) = \Psi(y)$, i.e. the restriction of the characters to $C^*(a)$ coincide. Since $\Phi(x) = \Phi(y)\,(= x_0)$, the characters also coincide on $\mathcal{A}$ and therefore are equal as characters in $\mathcal{B}$, since $\mathcal{B}$ is generated by $\mathcal{A}$ and $C^*(a)$.

On the other hand, if $x \in \Gamma(\mathcal{B})$ is such that $x(a) \neq 0$, then $\Phi(x) = x_0$. Indeed, assume that $\Phi(x) \neq x_0$. Let $f \in C(\Gamma(\mathcal{A}))$ with $f(\Phi(x)) = 0$ and $f(x_0) = 1$. So $f \circ \Phi \geq 1_{\Phi^{-1}(x_0)}$. But then

$$\int_{\Gamma(\mathcal{B})} f \circ \Phi \; dE_{\mathcal{B}} \geq \int_{\Gamma(\mathcal{B})} 1_{\Phi^{-1}(x_0)}\, dE_{\mathcal{B}} = E_{\mathcal{B}}(\Phi^{-1}(x_0) = E_{\mathcal{A}}(\{x_0\}) = p.$$

Note that if $0 \in \sigma(a)$ then it is an isolated point, so in any case we have $p \in C^*(a) \subseteq \mathcal{B}$. Then $0 = f \circ \Phi(x) \geq x(p) \geq 0$, so $x(p) = 0$. Since $0 \leq a \leq \beta\, p$, $x(a) = 0$ and the claim follows.

Now let $z \in \Phi^{-1}(x_0)$. If $z(a) \neq 0$, from the first part of the proof we deduce that $\Psi^{-1}(\Psi(z)) = \{z\}$. Therefore $E_{\mathcal{B}}(\{z\}) = E_a(\{\Psi(z)\}) = 0$, since $\Psi(z)(a) \neq 0$ and $\mathrm{At}(E_a) \subseteq \{0\}$. If $z(a) = 0$, then

$$\begin{aligned} \{z\} &= \Phi^{-1}(x_0) \setminus \{x \in \Phi^{-1}(x_0) :\; x(a) \neq 0\} \\ &= \Phi^{-1}(x_0) \setminus \Psi^{-1}(\{x \in \Gamma(C^*(a)) :\; x(a) \neq 0\}) \end{aligned}$$

and

$$\begin{aligned} E_{\mathcal{B}}(\Psi^{-1}(\{x \in \Gamma(C^*(a)) :\; x(a) \neq 0\})) &= E_a(\{x \in \Gamma(C^*(a)) :\; x(a) \neq 0\}) \\ &= E_{\mathcal{A}}(\{x_0\}) = E_{\mathcal{B}}(\Phi^{-1}(x_0)). \end{aligned}$$

From this we conclude that $E_{\mathcal{B}}(\{z\}) = 0$. □

Proof of Theorem 3.1. Recall that the set $\mathrm{At}(E_{\mathcal{A}})$ of atoms of $E_{\mathcal{A}}$ is a (possibly infinite) countable set. If $\mathrm{At}(E_{\mathcal{A}}) = \emptyset$ then $E_{\mathcal{A}}$ is already diffuse and we are done. Otherwise, let us enumerate $\mathrm{At}(E_{\mathcal{A}}) = \{x_i : 1 \le i \le r\}$, where $r \in \mathbb{N} \cup \{\infty\}$. For $1 \le i \le r$, let $I_i = [1+\frac{1}{2n}, 1+\frac{1}{2n-1}]$. Then $I_i \cap \overline{\bigcup_{1\le i\neq j\le r} I_j} = \emptyset$ and $\bigcup_{i=1}^r I_i \subseteq [1,2]$. For each $i = 1,\dots,r$ there exists, by Proposition 6.2, $a_i \in \mathcal{A}' \cap \mathcal{M}_{sa}$ such that $P_{\overline{R(a_i)}} = E_{\mathcal{A}}(\{x_i\})$, $I_i \subseteq \sigma(a_i) \subseteq I_i \cup \{0\}$, and such that $E_{\mathcal{A}_i}$ has no atoms in the fibre $\Phi_i^{-1}(x_i)$, where $\Phi_i : \Gamma(\mathcal{A}_i) \to \mathcal{A}$ denotes the continuous surjection induced by the inclusion $\mathcal{A} \subseteq \mathcal{A}_i := C^*(\mathcal{A}, a_i)$. Let $a = \sum_{i=1}^r a_i \in \mathcal{A}' \cap \mathcal{M}_{sa}$ (this sum converges because the ranges of the operators a_i are orthogonal and $\|a_i\| \le 2$ for every i). Then $\mathcal{B} := C^*(\mathcal{A}, a)$ is an abelian subalgebra of $\mathcal{M}$.

We claim that the spectral measure $E_{\mathcal{B}}$ of $\mathcal{B}$ has no atoms. Indeed, first note that $1_{I_i} \in C(\cup_{1\le j\le r} I_j)$ is a continuous function (because the distance between the sets I_i and $\cup_{i\neq j} I_j$ is positive); then, since $1_{I_i}(a) = a_i$, it follows that $\mathcal{A}_i \subset \mathcal{B}$ for every $i = 1,\dots,r$. Assume now that $x \in \mathrm{At}(\Gamma(\mathcal{B}))$ and let $\Phi : \Gamma(\mathcal{B}) \to \Gamma(\mathcal{A})$ be as before. By Lemma 6.1 there exists $i \in \{1,\dots,r\}$ such that $\Phi(x) = x_i \in \mathrm{At}(E_{\mathcal{A}})$. Since $\Phi = \Phi_i \circ \Psi_i$, where $\Psi_i : \Gamma(\mathcal{B}) \to \Gamma(\mathcal{A}_i)$ is the surjection induced by the inclusion $\mathcal{A}_i \subseteq \mathcal{B}$, we conclude that $\Psi_i(x) \in \Phi_i^{-1}(x_i)$ is an atom of the measure $E_{\mathcal{A}_i}$, again by Lemma 6.1. But this last assertion is a contradiction because by construction there are no atoms in the fibre $\Phi_i^{-1}(x_i)$ by construction. □

Remark 6.3. Given an abelian C^* subalgebra $\mathcal{A} \subset \mathcal{M}$, a direct way to find an abelian C^*-subalgebra $\mathcal{A} \subseteq \tilde{\mathcal{A}} \subset \mathcal{M}$ with diffuse spectral measure is to consider a masa in $\mathcal{M}$ that contains $\mathcal{A}$. The additional information we obtain from Theorem 3.1 is that $\tilde{\mathcal{A}}$ can be chosen separable (as a C^*-algebra) whenever $\mathcal{A}$ is separable. When this is the case, the character space of $\tilde{\mathcal{A}}$ is metrizable, a fact that is crucial for our calculations.

6.2. **Discrete approximations in separable diffuse abelian algebras.** Given a compact metric space it is always possible to find, using uniform continuity, discrete uniform approximations of continuous functions by linear combinations of characteristic functions of certain sets $\{Q_i\}_{i=1}^m$. But if we consider a measure on this space and we require equal measures for these sets, there might not be any good uniform approximation based on characteristic functions (even for measures of compact support in the real line). Proposition 3.2 is an intermediate solution to this problem. It was inspired by the proof of [13, Lemma 4.1].

Proof of Proposition 3.2. The space $\Gamma(\mathcal{B})$ is a metrizable compact topological space, so we consider a metric d in $\Gamma(\mathcal{B})$ inducing its topology. Let $r \in \mathbb{N}$; by compactness, there exists a partition $\{\tilde{Q}_i\}_{i=1}^{k_0}$ of $\Gamma(\mathcal{B})$ with $\mathrm{diam}_d(\tilde{Q}_i) < \frac{1}{r}$ and $\sum_{i=1}^{k_0} \mu_{\mathcal{B}}(\tilde{Q}_i) = 1$. Let $m = m(r)$ be such that $1/m \le \min\{\mu_{\mathcal{B}}(\tilde{Q}_j)^2 : 1 \le j \le k_0\}$. Then for $1 \le j \le k_0$ there exists $k_j \in \mathbb{N}$ such that $\mu_{\mathcal{B}}(\tilde{Q}_j) = k_j/m + \delta_j$ with $0 \le \delta_j < 1/m$. If we let $\tilde{k} = \tilde{k}(r) = \min_j\{k_j\}$ then $\tilde{k} \ge \max\{\mu_{\mathcal{B}}(\tilde{Q}_j)^{-1} - 1, 1 \le j \le k_0\}$.

For $t = 1,\dots,k_0$, choose $\tilde{k}$ partitions $\{\tilde{Q}^t_{j,s}\}_{s=0}^{k_j}$ of each $\tilde{Q}_j$ $(1 \le t \le \tilde{k})$, with $\mu_{\mathcal{B}}(\tilde{Q}^t_{j,s}) = 1/m$ if $1 \le s \le k_j$ and $\mu_{\mathcal{B}}(\tilde{Q}^t_{j,0}) = \delta_j$, in such a way that $\tilde{Q}^t_{j,0} \subset \tilde{Q}^1_{j,\,t}$, $2 \le t \le \tilde{k}$. Note that we can always make such a choice: using Lemma 2.4 choose $\tilde{Q}^t_{j,0} \subseteq \tilde{Q}^1_{j,\,t}$ with $\mu_{\mathcal{B}}(\tilde{Q}^t_{j,0}) = \delta_j < 1/m$, and then take a partition $\{\tilde{Q}^t_{j,s}\}_{s=1}^{k_j}$ of $\tilde{Q}_j \setminus \tilde{Q}^t_{j,0}$ using again Lemma 2.4 (note that $\mu_{\mathcal{B}}(\tilde{Q}_j \setminus \tilde{Q}^t_{j,0}) = k_j/m$). By this choice, $\tilde{Q}^t_{j,0} \cap \tilde{Q}^{t'}_{j,0} = \emptyset$ if $t \neq t'$.

For each $t = 1,\dots,\tilde{k}$, let $\tilde{Q}^t_{0,0} = \cup_{j=1}^{k_0} \tilde{Q}^t_{j,0}$. Then $\mu_{\mathcal{B}}(\tilde{Q}^t_{0,0}) = 1 - \sum_j k_j/m = (m - \sum_{j=1}^{k_0} k_j)/m$. Finally, make partitions of each set $\tilde{Q}^t_{0,0}$ into $n_1 = m - \sum_j k_j$ subsets $\{\tilde{Q}^t_i\}_{i=1}^{n_1}$ of measure $1/m$. By re-labeling the $\tilde{k}$ partitions $\{\tilde{Q}^t_{j,s}\}_{j,\,s} \cup \{\tilde{Q}^t_i\}_i$, we end up with $\tilde{k}$ partitions $\{Q_i^{t,\,m}\}_{i=1}^m$, for $1 \le t \le \tilde{k}$, such that:

1. $\mu_{\mathcal{B}}(Q_i^{t,\,m}) = 1/m$, for every $i \in \{1,\dots,m\}$, $t \in \{1,\dots,\tilde{k}\}$;

2. $\operatorname{diam}_d(Q_i^{t,m}) \le 1/r$, if $i > n_1$;
3. if $1 \le i,\, i' \le n_1$ then $Q_i^{t,m} \cap Q_{i'}^{t',m} = \emptyset$ if $i \ne i'$ or $t \ne t'$.

Note that the construction of the k partitions $\{Q_i^{t,m}\}_{i=1}^m$ was done in such a way that the subsets that do not have small diameters are disjoint, even for different partitions.

Let $\mathbb{M} = \{m(r),\, r \ge 1\}$ and for every $m = m(r) \in \mathbb{M}$ let $k(m) = \tilde{k}(r)$ as defined above and, for i, t, m, let $q_i^{t,m} = E_{\mathcal{B}}(Q_i^{t,m})$. The set $\mathbb{M}$ is unbounded because the measure $\mu_{\mathcal{B}}$ being diffuse makes $\lim_{r\to\infty} m(r) = \infty$, and so $\lim_{r\to\infty} \tilde{k}(r) = \infty$. For each $t = 1, \dots, k$, $\{q_i^{t,m}\}_{i=1}^m \subset \mathcal{B}' \cap \mathcal{M}$ is a partition of the unity.

Let $b \in \mathcal{B}$, $\epsilon > 0$, and let $f \in C(\Gamma(\mathcal{B}))$ be such that $b = \int_{\Gamma(\mathcal{B})} f\, dE_{\mathcal{B}}$. Then, by compactness, there exists $\delta > 0$ such that if $Q \subseteq \Gamma(\mathcal{B})$ with $\operatorname{diam}_d(Q) < \delta$ then $\operatorname{diam}(f(Q)) < \epsilon$. Let $r \in \mathbb{N}$ be such that $1/r < \delta$ and $2\|b\|/k(r) \le \epsilon$; let $m = m(r) \in \mathbb{M}$, and let $\beta_i^{t,m} = m\,\tau(b\, q_i^{t,m}) = m \int_{Q_i^{t,m}} f\, d\mu_{\mathcal{B}}$. Properties 1-3 translate then into

1'. $\tau(q_i^{t,m}) = 1/m$, for every $i \in \{1, \dots, m\}$, $t \in \{1, \dots, k\}$;
2'. if $i > n_1$, then $|f(x) - \beta_i^{t,m}| \le \epsilon$, $\forall x \in Q_i^{t,m}$;
3'. if $1 \le i,\, i' \le n_1$ then $q_i^{t,m} \perp q_{i'}^{t',m}$ if $i \ne i'$ or $t \ne t'$.

Therefore we have

$$\begin{aligned}
\left\| b - \frac{1}{k} \sum_{t=1}^{k} \sum_{i=1}^{m} \beta_i^{t,m}\, q_i^{t,m} \right\| &= \left\| \frac{1}{k} \sum_{t=1}^{k} \left(b - \sum_{i=1}^{m} \beta_i^{t,m}\, q_i^{t,m} \right) \right\| \\
&= \left\| \frac{1}{k} \sum_{t=1}^{k} \sum_{i=1}^{m} \int_{Q_i^{t,m}} (b - \beta_i^{t,m})\, dE_{\mathcal{B}} \right\| \\
&\le \left\| \frac{1}{k} \sum_{t=1}^{k} \sum_{i=1}^{n_1} \int_{Q_i^{t,m}} (f - \beta_i^{t,m})\, dE_{\mathcal{B}} \right\| + \epsilon \\
&\le \left\| \frac{2\,\|b\|}{k} \sum_{t=1}^{k} \sum_{i=1}^{n_1} q_i^{t,m} \right\| + \epsilon = \frac{2\|b\|}{k} + \epsilon \le 2\epsilon
\end{aligned}$$

where the first inequality is a consequence of 2' and the last equality follows from 3'. □

Proof of Lemma 3.5. Fix a norm dense subset $B = (b_j)_{j\in\mathbb{N}} \subseteq \mathcal{B}$. In the construction leading to Dixmier's Theorem, a previous result [17, 8.3.4] asserts that for each j, there exists a sequence $\{\rho_j^n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathcal{M})$ such that for every $1 \le h \le j$, $\|\rho_j^n(b_h) - \tau(b_h)\, I\| \xrightarrow{n} 0$. For each $j \in \mathbb{N}$, let $n_0 = n_0(j) \in \mathbb{N}$ be such that if $n \ge n_0$ then $\|\rho_j^n(b_h) - \tau(b_h)\, I\| \le 1/j$ for $1 \le h \le j$. If we let $\rho_j = \rho_j^{n_0(j)}$ for $j \in \mathbb{N}$, we get $\|\rho_j(b_h) - \tau(b_h)\, I\| \xrightarrow{j} 0$ for every $h \in \mathbb{N}$. Since $(b_j)_{j\in\mathbb{N}}$ is norm dense in $\mathcal{B}$ we have $\lim_j \|\rho_j(b) - \tau(b)\, I\| = 0$ for every $b \in \mathcal{B}$.

For every $i = 1, \dots, m$, consider the factor $p_i\mathcal{M}p_i$ with (normalized) trace $\tau_i(p_i x) = \tau(xp_i)/\tau(p_i)$. By the Dixmier approximation property mentioned in the first paragraph, applied to the separable C^*-subalgebra $p_i\mathcal{B}$ of the finite factor $p_i\mathcal{M}p_i$, there exists a sequence $\{\rho_j^i\}_{j\in\mathbb{N}} \in \mathcal{D}(p_i\mathcal{M}p_i)$ such that $\lim_{j\to\infty} \|\rho_j^i(p_i b) - \tau_i(p_i b) p_i\| = 0$, for every $b \in \mathcal{B}$.

For each $\rho \in \mathcal{D}(p_i\mathcal{M}p_i)$, we can consider an extension $\tilde{\rho} \in \mathcal{D}(\mathcal{M})$ as follows: if $\rho(p_i b) = \sum_{h=1}^k \lambda_h\; u_h\, b\, u_h^*$, with $u_h \in \mathcal{U}(p_i\mathcal{M}p_i)$, define $\tilde{\rho} \in \mathcal{D}(\mathcal{M})$ by $\tilde{\rho}(b) = \sum_{h=1}^k \lambda_h\; \tilde{u}_h\, b\, \tilde{u}_h^*$, where $\tilde{u}_h = u_h + (1 - p_i) \in \mathcal{U}(\mathcal{M})$. If $1 \le i \le m$ set $\rho_j = \prod_{i=1}^m \tilde{\rho}_j^i$ for $j \ge 1$. It is easy to verify that if $1 \le i \le m$ then $\rho_j(b\, p_i) = \tilde{\rho}_j^i(b\, p_i)$ for every $b \in \mathcal{B}$. Then, if $b \in \mathcal{B}$,

$$\left\| \rho_j(b) - \sum_{i=1}^{m} \beta_i(b) p_i \right\| = \left\| \sum_{i=1}^{m} \tilde{\rho}_j^{\,i}(b\, p_i) - \tau_i(b\, p_i) p_i \right\| \xrightarrow[j\to\infty]{} 0. \quad \square$$

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